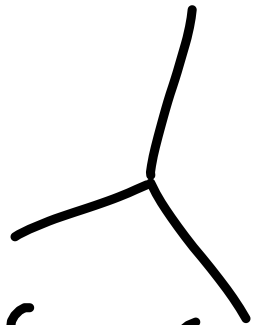
Betz limit proof

Assumptions: a) NO hub



b) wind is transverse

c) non-compressible

d) Uniform pressure

① mass flow rate $\frac{dm}{dt}$

$$\frac{dm}{dt} = \rho A_1 V_1 = \rho S v = \rho A_2 V_2$$

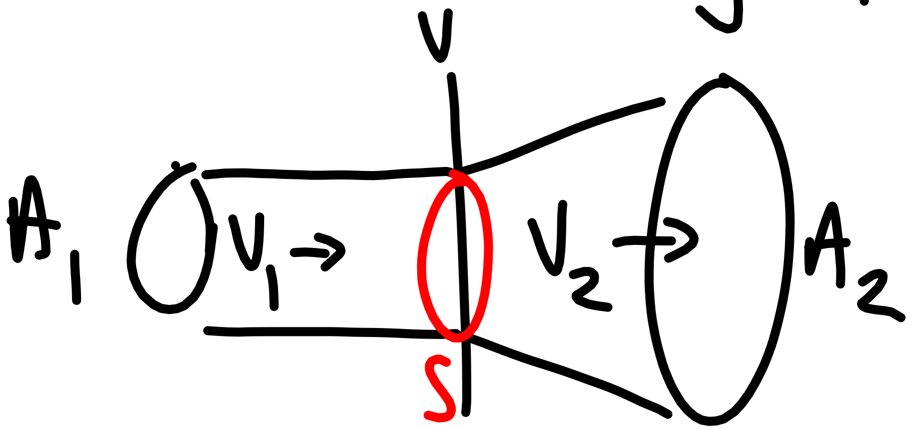
ρ = density-o-air

A_1 / A_2 = Area before/after

V_1 / V_2 = Velocity before/after

S = Area of turbine

V = velocity of wind through turbine



Units: $\rho A_1 V_1$

$$\frac{kg}{\cancel{m^3}} \cdot \frac{\cancel{m^2}}{1} \cdot \frac{\cancel{m}}{s} = \frac{kg}{s} \checkmark$$

(2) Force on turbine

$$F = ma$$

$$F = m\frac{dv}{dt} \quad \underline{or} \quad \frac{m}{dt}dv$$

$$F = \boxed{\frac{dm}{dt}}dv$$

$$F = \rho S v dv \quad \underline{or} \quad F = \rho S v(v_1 - v_2)$$

③ Power of turbine

$$P = \frac{W}{t} = \frac{\vec{F}\cdot\vec{s}}{t} = \vec{F}\frac{ds}{dt} = \boxed{\vec{F}}\,\vec{v}$$

$$P = \underbrace{\rho S v (v_1 - v_2)}_{F}\, v$$

$$P = \rho S v^2 (v_1 - v_2)$$

But : work = ΔE

So: $P = \dfrac{\Delta E}{\Delta t} = \dfrac{K_1 - K_2}{\Delta t}$ $\frac{1}{2}mv^2$

$$P = \frac{1}{2}\frac{dm}{dt}(v_1^2 - v_2^2)$$

$$P = \frac{1}{2}\rho S v (v_1^2 - v_2^2)$$

$(v_1 - v_2)(v_1 - v_2)$

$v_1^2 - v_2^2$

$$\cancel{\rho}\cancel{S}v^{\cancel{2}}(v_1 - v_2) = \frac{1}{2}\cancel{\rho}\cancel{S}\cancel{v}(v_1^2 - v_2^2)$$

$$v(v_1 - v_2) = \frac{1}{2}(v_1^2 - v_2^2)$$

$$v(\cancel{v_1 - v_2}) = \frac{1}{2}(\cancel{v_1 - v_2})(v_1 + v_2)$$

$$v = \frac{1}{2}(v_1 + v_2)$$

4 Put it together

$$P = \frac{1}{2}\rho S \boxed{v} (v_1^2 - v_2^2)$$

$$P = \frac{1}{2}\rho S \left(\frac{1}{2}v_1 + v_2\right)(v_1^2 - v_2^2)$$

$$P = \frac{1}{4}\rho S (v_1 + v_2)(v_1^2 - v_2^2)$$

From graph: $\frac{1}{3} = \frac{v_2}{v_1}$ is best

$$P=\frac{1}{4}\rho S\left(v_1+v_2\right)\left(v_1^2-v_2^2\right)$$

$$P=\frac{1}{4}\rho S\left(v_1^3-v_1v_2^2+v_1^2v_2-v_2^3\right)$$

$$P=\frac{1}{4}\rho S\,v_1^3\left(\frac{v_1^3}{v_1^3}-\frac{\cancel{v_1}v_2^2}{v_1^{\cancel{3}}}+\frac{v_1^{\cancel{2}}v_2}{v_1^{\cancel{3}}}-\frac{v_2^3}{v_1^3}\right)$$

$$P=\frac{1}{4}\rho S v_1^3\left(1-\underbrace{\left(\frac{v_2}{v_1}\right)}_{\frac{1}{3}}{}^2+\underbrace{\left(\frac{v_2}{v_1}\right)}_{\frac{1}{3}}-\underbrace{\left(\frac{v_2}{v_1}\right)}_{\frac{1}{3}}{}^3\right)$$

$$P=\frac{1}{4}\rho S v_1^3\left(\frac{32}{27}\right)$$

$$P_{max} = \frac{1}{2} \rho S v_1^3 \left(\frac{16}{27}\right)$$

$\to$ 59.3%

$\to C_{p\,max}$